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Instability of rigidly rotating flows to non-axisymmetric disturbances

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An investigation of the hydrodynamic stability of swirling flows having arbitrary Rossby numbers is described. A necessary condition for instability is derived for rigidly rotating flows and this condition is further refined in the specific case of a parabolic axial flow. Numerical results are presented for two azimuthal wave-numbers corresponding to the maximum growth rates of unstable perturbations as a function of Rossby number. It is found that the largest growth rates occur when the Rossby number is $O(1)$ and that instability persists for surprisingly large values of this parameter. Previous explanations of the instability mechanism are discussed and it is concluded that these are only adequate in the limit of small Rossby number.

1. Introduction

One usually thinks of rotation as being a stabilizing influence on shear flows because of the analogy that exists between rotating and stratified flows. Thus, for *axisymmetric* disturbances, a swirl component $V(r)$ acts upon an axial shear flow $W(r)$ in much the same manner as would a radial gravitational field. Howard & Gupta (1962), in fact, showed that an equivalent 'local Richardson number' could be defined for such a swirl flow and stability assured if this quantity was everywhere greater than $\frac{1}{4}$. Significantly, they were *not* able to determine such a stability criterion for non-axisymmetric disturbances.

Subsequently, Pedley (1968) investigated the small Rossby number limit (rapid rotation) for the rigidly rotating pipe flow with velocity components

$$\{0, \Omega_0^* r, W_0[1-(r/r_0)^2]\} \tag{1.1}$$

in cylindrical co-ordinates $\{r, \theta, z\}$. In that limit, he found that the growth rate of the most unstable helical perturbations was proportional to the Rossby number ϵ defined by

$$\epsilon = W_0/\Omega_0^* r_0. \tag{1.2}$$

This interesting result is suggestive, but not conclusive, in regard to the possible breakdown of laminar flows because of the assumption that $\epsilon \ll 1$. What one would like to know, especially from a practical standpoint, is what happens when ϵ is larger. (Clearly, the growth rate attains a maximum at some value of ϵ and then decays to zero as $\epsilon \to \infty$, because circular Poiseuille flow without rotation is stable.) This question is answered by the numerical results presented in § 5, which show that the instability does turn out to be very powerful at finite values of the Rossby number.

The instability of swirling flows has also been studied by Scorer (1967), who employed a 'localized' analysis. It is difficult to relate Scorer's results to those of the normal-mode approach, but instability is predicted for the rotating pipe flow and interestingly, as shown in § 6, the predicted direction of maximum instability coincides with the corresponding result by Pedley when $\epsilon \ll 1$. The instability mechanism, as described by Scorer, is the well-known centrifugal instability when viewed in an appropriate co-ordinate system. A very similar argument was given in the appendix of a second paper by Pedley (1969), dealing with the viscous case. In the latter paper, the Rayleigh circulation criterion was extended to include flows having a small axial component in the direction of the rotation axis. It appears that these ideas can be applied quite successfully in flows having *small* Rossby numbers. However, the necessary condition for instability derived below, which is valid at arbitrary Rossby numbers, is not related to Rayleigh's theorem. As discussed in § 6, the instability mechanism is evidently much more complicated in the general case of finite ϵ.

2. Necessary conditions for instability

The radial perturbation velocity for non-axisymmetric disturbances has the form

$$\hat{u} = u(r)\exp\{i(kz+m\theta-\omega t)\}, \tag{2.1}$$

where ω is complex. A single second-order differential equation has been derived by Howard & Gupta (their equation (18)) for $u(r)$. This equation in the special case of rigid rotation takes the form

$$\gamma^2\frac{d}{dr}\left\{\frac{S}{r}\frac{d}{dr}(ru)\right\}-\left\{\gamma^2+\gamma r\Omega_0\frac{d}{dr}\left[\frac{S}{r}\left(k\epsilon\frac{dW}{dr}+\frac{2m}{r}\right)\right]-2k\Omega_0^2\frac{S}{r}\left(2kr-\epsilon m\frac{dW}{dr}\right)\right\}u = 0, \tag{2.2}$$

where

$$S = r^2(m^2+k^2r^2)^{-1}, \quad \gamma = k\epsilon\Omega_0 W(r)+m\Omega_0-\omega. \tag{2.3}$$

All quantities have been non-dimensionalized with respect to a characteristic length, velocity and frequency denoted by r_0, W_0 and Ω, respectively. The characteristic frequency is defined by

$$\Omega = [\tfrac{1}{2}(\Omega_0^{*2}+W_0^2/r_0^2)]^{\frac{1}{2}} = \Omega_0^*[\tfrac{1}{2}(1+\epsilon^2)]^{\frac{1}{2}}, \tag{2.4}$$

where ϵ, the Rossby number, is as defined in (1.2). The quantity Ω is an appropriate characteristic frequency for *all* values of ϵ; i.e. in the limit $\epsilon \ll 1$, $2^{\frac{1}{2}}\Omega \sim \Omega_0^*$, while for $\epsilon \gg 1$, $2^{\frac{1}{2}}\Omega \sim W_0/r_0$. When $\epsilon = 1$, $\Omega = \Omega_0^* = W_0/r_0$. Note also that $\Omega_0 = \Omega_0^*/\Omega = \{\tfrac{1}{2}(1+\epsilon^2)\}^{-\frac{1}{2}}$ in (2.2) and (2.3).

If we now follow the approach of Howard & Gupta and make the substitution, with $\omega_i > 0$,

$$u = H\gamma^{1-n}, \tag{2.5}$$

equation (2.2) can be put in the form

$$\frac{d}{dr}\left\{\gamma^{2(1-n)}\frac{S}{r}\frac{d}{dr}(rH)\right\}-\gamma^{2(1-n)}\left\{1+\frac{r\Omega_0}{\gamma}\frac{d}{dr}\left[\frac{S}{r}\left(nk\epsilon\frac{dW}{dr}+2\frac{m}{r}\right)\right]\right.$$
$$\left.+\frac{S\Omega_0^2}{\gamma^2}\left[n(1-n)\left(k\epsilon\frac{dW}{dr}\right)^2-2k\left(2k-\frac{\epsilon m}{r}\frac{dW}{dr}\right)\right]\right\}H = 0. \tag{2.6}$$

Noting that $dS/dr = 2m^2S^2/r^3$, we can rewrite the above equation as

$$\frac{d}{dr}\left\{\gamma^{2(1-n)}\frac{S}{r}\frac{d}{dr}(rH)\right\} - \gamma^{2(1-n)}\left\{1+\frac{kr\Omega_0}{\gamma}\left[\epsilon n\frac{d}{dr}\left(\frac{S}{r}\frac{dW}{dr}\right)-\frac{4mkS^2}{r^3}\right]\right.$$
$$\left.+\frac{S\Omega_0^2}{\gamma^2}\left[n(1-n)\left(k\epsilon\frac{dW}{dr}\right)^2 - 2k\left(2k-\frac{\epsilon m}{r}\frac{dW}{dr}\right)\right]\right\}H = 0. \quad (2.7)$$

We first set $n = 0$, then multiply (2.7) by $r\bar{H}$, where the bar denotes the complex conjugate. Integrating now between the boundaries, say r_1 and r_2 in the case of coaxial cylinders, we obtain

$$-\int_{r_1}^{r_2}\gamma^2\left[\frac{S}{r}\left|\frac{d}{dr}(rH)\right|^2 + r|H|^2\right]dr + 4mk^2\Omega_0\int_{r_1}^{r_2}\gamma S^2 r^{-1}|H|^2\,dr$$
$$+2k\Omega_0^2\int_{r_1}^{r_2}S\left(2k-\frac{\epsilon m}{r}\frac{dW}{dr}\right)r|H|^2\,dr = 0. \quad (2.8)$$

If we write $\gamma = \gamma_r + i\gamma_i$, the imaginary part of (2.8) yields, after multiplication by m,

$$\gamma_i\left\{-2\int_{r_1}^{r_2}m\gamma_r\left[\frac{S}{r}\left|\frac{d}{dr}(rH)\right|^2 + r|H|^2\right]dr + 4m^2k^2\Omega_0\int_{r_1}^{r_2}\frac{S^2}{r}|H|^2\,dr\right\} = 0. \quad (2.9)$$

Because both the quantity in square brackets and the second integral are always positive, we find that the quantity $m\gamma_r$ *must be positive* somewhere in (r_1, r_2) for instability ($\gamma_i \neq 0$) to occur. (The convention that will be employed is that k is always positive; however, both m and ω_r can take either sign.)

Next, we set $n = 1$ in (2.7) and again multiply by $r\bar{H}$ and integrate between the boundaries to obtain

$$-\int_{r_1}^{r_2}\left[\frac{S}{r}\left|\frac{d}{dr}(rH)\right|^2 + r|H|^2\right]dr - k\Omega_0\int_{r_1}^{r_2}\frac{r^2}{\gamma}\left[\epsilon\frac{d}{dr}\left(\frac{S}{r}\frac{dW}{dr}\right)-\frac{4kmS^2}{r^3}\right]|H|^2\,dr$$
$$+2k\Omega_0^2\int_{r_1}^{r_2}\frac{S}{\gamma^2}\left(2k-\frac{\epsilon m}{r}\frac{dW}{dr}\right)r|H|^2\,dr = 0. \quad (2.10)$$

In the particular case of fully developed pipe flow, $W = 1-r^2$, and the imaginary part of (2.10) can be written as

$$-4k\Omega_0\gamma_i\int_0^1 S(k+\epsilon m)\left(\frac{m^2S}{r^2|\gamma|^2}+\frac{2\Omega_0 m\gamma_r}{|\gamma^2|^2}\right)r|H|^2\,dr = 0. \quad (2.11)$$

To arrive at this result, (2.10) has been multiplied by m and dS/dr expressed in terms of S. Since the constant term $(k+\epsilon m)$ is in general non-zero, we see that for instability $m\gamma_r$ *must be negative* somewhere in (r_1, r_2).

Thus, in the case of rigidly rotating pipe flow, it has been proved that a *necessary condition for instability* is that $\gamma_r = 0$ somewhere in $(0, 1)$. One physical interpretation of this result (there are others) is the following: in a frame of reference moving with the *axial* phase speed of the wave, the mean flow velocity vector can be written as

$$\mathbf{q} = (\Omega_0\epsilon W - \omega_r/k)\,\mathbf{e}_z + \Omega_0 r\mathbf{e}_\theta, \quad (2.12)$$

where $\mathbf{q}$ has been non-dimensionalized with respect to Ωr_0. The wavenumber vector $\mathbf{k}$ is given by the gradient of a constant-phase surface, i.e.

$$\mathbf{k} = \nabla\phi = k\mathbf{e}_z + (m/r)\,\mathbf{e}_\theta, \tag{2.13}$$

where $\phi = kz + m\theta - \omega_r t$ and $\mathbf{k}$ is dimensionless. It can now be seen that $\gamma_r = \mathbf{k}.\mathbf{q}$, so that for instability to occur the mean velocity component in the direction of propagation of an unstable perturbation must vanish at some value of r. Alternatively, we can consider a fixed value of r and note that the intersection of a circular cylinder with a constant-phase surface is a helical curve and the foregoing result states that the velocity component normal to this helix must vanish.

Finally, by setting $n = \frac{1}{2}$ in (2.7), integrating between the boundaries and taking the imaginary part of the resulting equation, an upper bound on the growth rate of unstable perturbations can be obtained. This was done by Howard & Gupta and the dimensionless equivalent of their equation (22) takes the form

$$\omega_i^2 \leqslant \max\left\{\frac{4\Omega_0^2 r^2}{(m/k)^2+r^2}\left[\left(\frac{\epsilon r}{2}\right)^2 - 1 - \frac{\epsilon m}{k}\right]\right\}, \tag{2.14}$$

in the case of rigidly rotating pipe flow. This result is very important because it shows that, if instability is directly related to the perturbation being non-axisymmetric ($m \neq 0$), then m has to be negative. As pointed out by Dr T. J. Pedley, the quantity on the right-hand side of (2.14) reaches its maximum at $r = 1$, so that, in the unstable case,

$$\omega_i^2 \leqslant \Omega_0^2\left[\frac{\epsilon^2 - 4\epsilon m/k - 4}{1+(m/k)^2}\right]. \tag{2.15}$$

Thus, a further necessary condition for instability is that

$$-m/k > (4-\epsilon^2)/4\epsilon. \tag{2.16}$$

3. Solution in the limit of small Rossby number

The results obtained in this section are equivalent to those derived previously by Pedley (1968), whose approach was to reduce the governing equation to Sturm–Liouville form. Here, it is shown how the results of § 2 can be employed to arrive more directly at the same solution.

First, consider the inequality (2.14) and note that, in the case $\epsilon \ll 1$, the quantity in square brackets can only be positive if k/m is $O(\epsilon)$ or smaller. However, if we take $k/m \ll 1$, the maximum possible growth rate will be proportional to k/m; this suggests that to obtain maximum instability we take $k/m \sim O(\epsilon)$ and not less. Combining this observation with the result proved above, that $\gamma_r = 0$ somewhere in the unstable case, we see that

$$\gamma_r \simeq m\Omega_0 - \omega_r = 0, \tag{3.1}$$

because the first term in γ_r (see (2.3)) is $O(\epsilon^2)$. Equation (3.1), in conjunction with the observation that m will be negative in the unstable case, tells us that growing modes spiral in the *same* direction as the basic flow rotation but propagate *upstream* in the axial direction with an axial phase speed $O(\epsilon^{-1})$.

Employing the approximations suggested above, we now have $S \simeq r^2/m^2$ and $\gamma \simeq -i\omega_i$, and (2.2) becomes, in the case $W = 1-r^2$,

$$\frac{d^2}{dr^2}(ur)+\frac{1}{r}\frac{d}{dr}(ur)-\left[\frac{4k(k+\epsilon m)\,\Omega_0^2}{\omega_i^2}+\frac{m^2}{r^2}\right](ur) = 0. \tag{3.2}$$

This is simply Bessel's equation of order m, so that the solution for $u(r)$ that is bounded at $r = 0$ is given by

$$u(r) = r^{-1}J_m(\lambda r), \quad \text{where} \quad \lambda^2 = -4k\Omega_0^2(k+\epsilon m)/\omega_i^2, \tag{3.3}$$

and J_m is a Bessel function of the first kind. Permissible values of λ are those coinciding with the zeros of the Bessel function as the boundary condition $u(1) = 0$ is then satisfied. These values can be found readily from tables because m, the azimuthal wavenumber, is always an integer.

Rewriting (3.3) as $\omega_i^2 = -4k\Omega_0^2(k+\epsilon m)/\lambda^2$, we see that the fastest-growing waves correspond to the first zero, i.e. the smallest value of λ. The largest (absolute) values of m lead to the largest growth rates and, for specified values of m and ϵ, the axial wavenumber leading to the largest growth rate is

$$k = -\tfrac{1}{2}\epsilon m. \tag{3.4}$$

Additionally, it can now be verified that, for the most unstable perturbations, $\omega_i \sim O(\epsilon)$ as suggested by (2.14). By employing (3.4) and noting from (2.4) that $\Omega_0 \simeq 2^{\frac{1}{2}}$, we obtain finally

$$\omega_{i\,\max} = 2^{\frac{1}{2}}\epsilon|m|/\lambda. \tag{3.5}$$

Recall, however, that the foregoing results are limited to inviscid flow with $\epsilon \ll 1$. At finite Reynolds numbers, the damping effect of viscosity will be greatest for short waves, so the conclusion that the largest values of m correspond to the largest growth rates will be modified. In fact, it was found by Pedley (1969) that, as the Reynolds number increases, the first mode to be destabilized is the $m = -1$ mode, this occurring at a Reynolds number of 82·9. Joseph & Carmi (1969) also obtained approximately the same value using the energy method. The most notable deviations from the results of Pedley (1968), however, are those occurring at finite values of the Rossby number, as shown below.

4. Numerical procedure in the general case

The equation to be solved is (2.2), which, in the case $W = 1-r^2$, takes the form

$$\frac{d^2u}{dr^2}+\frac{1}{r}\left(1+\frac{2m^2S}{r^2}\right)\frac{du}{dr}-\left\{\frac{1}{S}\left(1+\frac{S}{r^2}-\frac{2m^2S^2}{r^4}\right)-\frac{4k\Omega_0}{\gamma^2}(k+\epsilon m)\left(\Omega_0+\frac{\gamma mS}{r^2}\right)\right\}u = 0, \tag{4.1}$$

with the boundary conditions

$$\left.\begin{aligned} u(0) = u(1) = 0 \quad &\text{when} \quad |m| > 1,\\ u'(0) = u(1) = 0 \quad &\text{when} \quad |m| = 1.\end{aligned}\right\} \tag{4.2}$$

The point $r = 0$ requires special consideration owing to the regular singularity occurring there. Expanding about this point by the method of Frobenius, we find that the solution bounded at $r = 0$ behaves for small r like

$$u = Ar^{|m|-1}(1 + a_2 r^2 + \ldots), \tag{4.3}$$

where $$a_2 = \frac{k}{4(|m|+1)}\left[k + \frac{2k}{m} - \frac{4\Omega_0}{\{\gamma(0)\}^2}(\epsilon m + k)\left(\Omega_0 + \frac{\gamma(0)}{m}\right)\right].$$

Because $u(0) = u'(0) = 0$ according to (4.3) (except when $m = -1$), it is necessary to begin the integration at a finite value of r. It was found that satisfactory results were obtained by starting the integration at $r = 0{\cdot}10$ or $r = 0{\cdot}20$. The procedure employed was to compute starting values for u and u' from (4.3) and then to integrate (4.1) out to $r = 1$ using a fourth-order Runge–Kutta procedure and double-precision complex arithmetic.

In the eigenvalue problem associated with (4.1)–(4.3), all but two of the various parameters can be specified independently. Accordingly, k, ϵ, m and Ω_0 were selected in advance; a two-variable Newton–Raphson iteration subroutine was then employed to find those values of ω_r and ω_i for which the boundary condition $u(1) = 0$ was satisfied. This represents, of course, two conditions because both the real and imaginary part of $u(1)$ must vanish.

The computations were begun at small values of ϵ in order that the solution (3.1)–(3.5) could be used to provide starting guesses for ω_r and ω_i. As ϵ was increased, previous solutions were used to estimate the magnitude of ω_i, while ω_r could be determined closely from the condition that $\gamma_r = 0$ somewhere in $(0, 1)$.

An interesting numerical difficulty that arises in this problem is associated with the multiple zeros of the eigenfunction. In the limit $\epsilon \ll 1$, $u(r)$ can be taken to be real and from (3.3) it is clear that the lowest zero of the Bessel function will lead to the largest growth rate. It seems likely that with $\epsilon \sim O(1)$ the most unstable mode will still correspond to the 'first zero'. However, $u(r)$ is now complex and solutions exist, for example, in which the condition $u(1) = 0$ is satisfied by the first zero of u_r and the second zero of u_i. In many cases, such a solution can be converted into one where the lowest mode does correspond to the first zero for both u_r and u_i by a suitable choice of the arbitrary constant multiplying u. (It was found that setting $A = 1 + 0{\cdot}26i$ in (4.3) worked well for the parameter range investigated.) However, it was not always possible to do this; therefore, ϵ had to be increased slowly in making the numerical computations in order to ensure that the same solution was being followed continuously.

5. Numerical results

Maximum growth rates have been determined as a function of the Rossby number ϵ for two values of m, these being $m = -4$ and $m = -1$. As noted in § 3, growth rates increase with $|m|$ in the small-ϵ case. However, the numerical results showed little variation with m for $|m| \geqslant 2$ at finite values of ϵ; hence, apart from the qualitatively different case $m = -1$, only the results for $m = -4$ are presented here as these are quite representative.

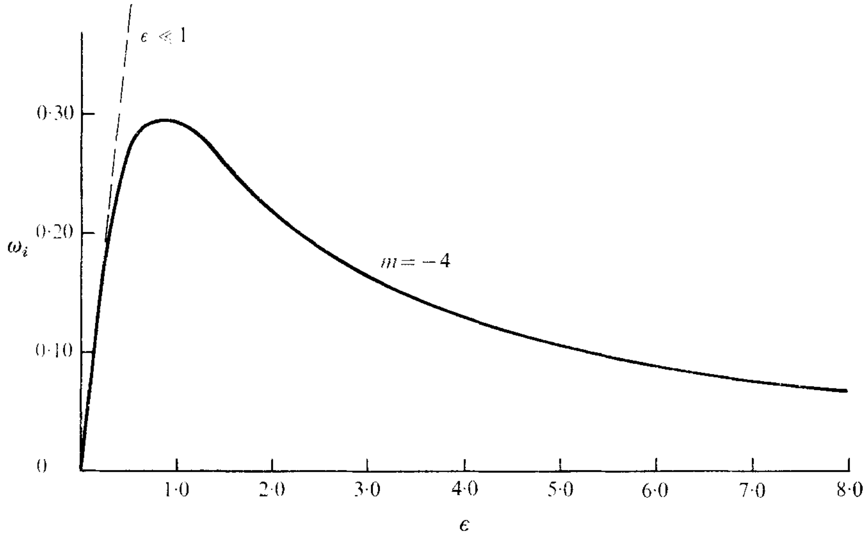

FIGURE 1. Maximum amplification factor *vs.* Rossby number for an azimuthal wavenumber $m = -4$.

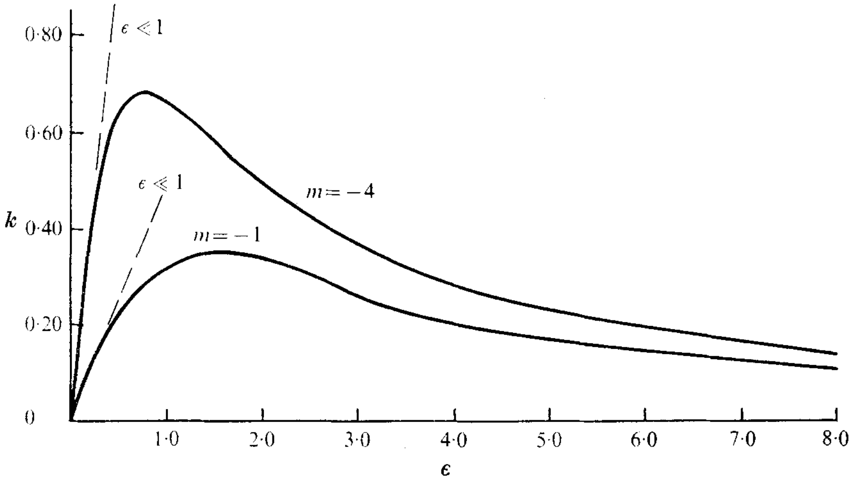

FIGURE 2. Axial wavenumber for fastest-growing waves as a function of Rossby number with m = constant.

In figure 1, the maximum values of ω_i are shown as a function of ϵ. It is seen that the largest growth rate of $\omega_i = 0{\cdot}296$ occurs at $\epsilon = 0{\cdot}80$. The most striking features of the results in this figure are the strength of the instability at $O(1)$ Rossby number as measured by the magnitude of ω_i and its persistence at very large values of ϵ. Shown for comparison is the prediction of the small-ϵ analysis as given by (3.5).

For each value of ϵ, with m held constant, a range of values of k was considered in order to find the axial wavenumber corresponding to the largest growth rate. These results are illustrated in figure 2 for the cases $m = -1$ and

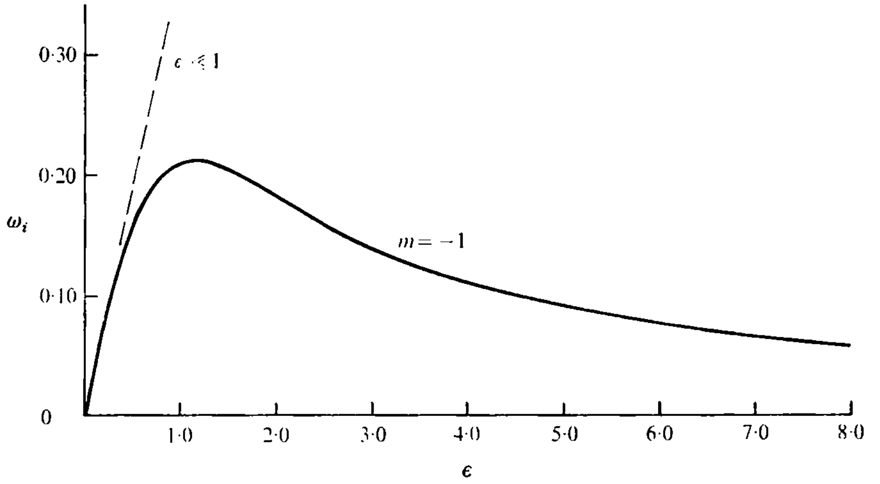

FIGURE 3. Maximum amplification factor *vs.* Rossby number for the sinuous mode, $m = -1$.

m	ϵ	k	ω_r	ω_i
−1	0·5	0·22	−1·183	0·1542
−1	1·0	0·32	−0·815	0·2083
−1	2·0	0·34	−0·395	0·1818
−1	4·0	0·20	−0·199	0·1109
−1	6·0	0·14	−0·132	0·0770
−1	8·0	0·11	−0·096	0·0586
−4	0·5	0·64	−4·876	0·2636
−4	1·0	0·66	−3·715	0·2938
−4	2·0	0·49	−2·273	0·2176
−4	4·0	0·28	−1·225	0·1247
−4	6·0	0·20	−0·817	0·0854
−4	8·0	0·15	−0·616	0·0647

TABLE 1. Eigenvalues for the most unstable waves

$m = -4$. The value of k corresponding to a particular value of ϵ in figure 1 is found from the $m = -4$ curve in figure 2 at the same value of ϵ. Again the prediction of the analysis with $\epsilon \ll 1$, i.e. equation (3.4), is shown for comparison.

The second case studied in detail, $m = -1$, is of particular interest for two reasons. First of all, it is the 'sinuous mode' in which the radial velocity at the centre of the pipe is non-zero (but continuous). The eigenfunction, as a result, has a considerably different structure with the maximum perturbation energy occurring at $r = 0$. Second, at finite Reynolds numbers, it seems likely that this mode will have the lowest 'critical Reynolds number' for all values of ϵ. (This conclusion seems to be supported by the viscous calculations of Mackrodt (1973), which were recently called to the present author's attention by Professor P. G. Saffman.)

The numerical results for ω_i are shown in figure 3, which has the same qualitative behaviour as figure 1. For small values of ϵ the growth rates of the sinuous mode are noticeably less than those of the $m = -4$ mode, but the difference between the two decreases as ϵ rises. At $\epsilon = 2$, for example, ω_i is only 16 % lower when $m = -1$. The largest growth rate occurs at $\epsilon = 1{\cdot}25$, this being $\omega_i = 0{\cdot}210$.

Some of the computed eigenvalues are given in table 1. Note that in all cases ω_r is negative, so that the unstable waves propagate upstream in the axial direction, although more slowly than in the small-ϵ limit. This characteristic is somewhat reminiscent of a result proved by Acheson (1972), namely that unstable non-axisymmetric hydromagnetic waves must propagate westward in the azimuthal direction in the case of flow in a rotating annulus.

6. Discussion

The primary result of the numerical calculations is that the growth rate of unstable helical perturbations is very large when the Rossby number is $O(1)$. Previously, Pedley (1968) had shown that an amplification factor of $O(\epsilon)$ was possible in the limiting case $\epsilon \ll 1$. Such instabilities do not necessarily ensure transition to turbulence because the effects of viscosity and nonlinearity are damping; i.e. a finite amplitude equilibrium state is likely for modes whose linear amplification rates are not substantial. However, the growth rates illustrated in figures 1 and 3 are so large that there can be no doubt that the instability mechanism associated with non-axisymmetric disturbances is a powerful one.

Also of interest is the persistence of the instability for $\epsilon \gg 1$ and the rapid variation of ω_i with ϵ when ϵ is $O(1)$. This undoubtedly is an important part of the explanation for the poor quantitative agreement between the results of Pedley (1969) and the experiments of Nagib *et al.* (1971). It appears that the Rossby numbers in these experiments ranged from $O(1)$ to very large values.

The approximate solution for $\epsilon \ll 1$ is seen in figures 1–3 to deviate markedly from the numerical solution of the full equation when ϵ is greater than 0·3. This seems, at first, surprising because (3.2) is an accurate approximation (to $O(\epsilon^2)$) of the full equation (4.1). However, the failure of the approximation is actually associated with the result $\gamma_r = 0$ stated in (3.1). Because $m\Omega_0 - \omega_r = 0$ in the solution for $\epsilon \ll 1$, the neglected term, $k\Omega_0 \epsilon W$, although $O(\epsilon^2)$, is really not all that small in comparison with $m\Omega_0 - \omega_r$. The results of the solution are, of course, still valuable because they reveal the existence of the instability and provide starting values for the numerical computations.

The matter of primary interest is to gain an understanding of the mechanism of this instability. In the appendix of the paper by Pedley (1969), it was argued on the basis of some flows qualitatively similar to the rotating pipe flow that the mechanism of instability was simply the centrifugal instability first explained by Rayleigh. (Here, we shall avoid the term 'inertial instability' employed by Pedley in order to prevent confusion with the non-rotating case.) In the examples discussed by Pedley, the Rayleigh circulation criterion was extended to swirl flows by restating the result as being that instability occurred if the mean vorticity component *in the direction of the wavenumber vector* was negative at some value of r.

That argument, with some modification, can also be applied to the rotating pipe flow with an interesting result. To adapt Rayleigh's theorem in the present case, recall that in rotating Couette flow, for example, the axial phase speed of unstable perturbations is zero. Here, let us consider the flow in a system propagating in the z direction at the speed of the wave so that the apparent axial phase speed is zero. The non-dimensional vorticity is given by

$$\nabla \times \mathbf{q} = 2\Omega_0(\mathbf{e}_z + \epsilon r \mathbf{e}_\theta) \tag{6.1}$$

and, if we now apply the modified Rayleigh criterion as stated by Pedley, we find that

$$\mathbf{k}.(\nabla \times \mathbf{q}) = 2\Omega_0(k + \epsilon m) < 0 \tag{6.2}$$

for instability to occur. If we compare (6.2), which is a result apparently not restricted to the case of small ϵ, with (3.3) it is clear that satisfaction of the criterion (6.2) does, in fact, lead to instability in the small-ϵ case.

It seems unlikely that this result is purely coincidental; on the other hand it also is very doubtful that such a simple argument will work in general when ϵ is finite. For one thing, the necessary condition for instability derived in § 2 *is* applicable in the general case and does not seem to be directly related to (6.2). (It must be conceded though that there were no cases of instability in the numerical calculations where (6.2) was violated.) One reason for regarding (6.2) with suspicion, however, is that this type of reasoning seems to lead to a conclusion in the local stability analysis of Scorer (1967) that is correct for small ϵ, but grossly in error for finite values of ϵ. Scorer considered the generation of unstable vortical motions for flows of the type considered here by analysing locally the effect of rotational displacements of a fluid 'parcel'. According to Scorer's analysis, the direction of maximum instability in a helical flow is along the bisector of the axis of the helix and the mean flow vorticity vector.

We can translate this statement into the terms of a normal-mode analysis in the following way: according to (6.1), the angle between the axis of the cylinder and the vorticity vector is given by

$$\delta = \tan^{-1} \epsilon r. \tag{6.3}$$

The axis of an unstable vortex, if the vortices are spaced periodically, makes an angle

$$\beta = \tan^{-1} \frac{k}{(-m/r)} \tag{6.4}$$

with the cylinder axis, assuming m is negative. Scorer's result states that $\beta = \frac{1}{2}\delta$, which for $\epsilon \ll 1$ and $k = O(\epsilon)$ yields the result

$$k = -\tfrac{1}{2}\epsilon m, \tag{6.5}$$

which is identical to (3.4). This interesting correspondence between the results of Scorer and Pedley does not seem to have been noticed previously. Scorer, however, goes on to predict that β increases continuously with ϵ and that in the large Rossby number limit $\beta = 45°$. This is clearly incorrect in the case studied here as can be seen from figure 2, where $k/(-m)$ does not increase indefinitely with ϵ.

This failure of the localized analysis is not too surprising because such an approach is only valid for large wavenumbers. While the notion of modifying Rayleigh's criterion is appealing, it seems that the real situation in the general case is more complicated than that. The necessary condition for instability derived here seems to shed little light on the physical mechanism, although it is certainly suggestive because singular neutral modes are often associated with instability. In that respect, the effect of rotation is somewhat similar to the influence of stable density stratification or viscosity, where an apparently stabilizing influence leads to an instability through some mechanism that is only vaguely understood.

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